A study of the static properties of a mass-spring network with a spider-web topology

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February 28, 1998

In this document, we study the behaviour of a particular spring-mass network, without any consideration for any reference object. However in the end of this document, we will mention a concrete case (droplet deformations [1]) in which this analytical work was used.

1 The spider-web spring-mass network

1.1 A *level/slice* structure

A spider-web spring-mass model is composed of $NM + 1$ masses. These masses are grouped in $N + 1$ sets called *levels*. The levels can be numbered from 1 to $N + 1$. Level number one is called the *basis level* and level number $N + 1$ is called the *apex*. Each level (i.e. each set of masses) contains M masses, except level $N + 1$ (the *apex*) which contains only one mass (the *apex* mass).

The masses on levels 1 to N are also numbered from 1 to M. The masses which are on different levels and have the same number within the loop are said to be on the same *slice*.

Apart from the apex, each mass is characterized by two numbers $i \in [1, N]$ and $j \in [1, M]$. i is the number of the level on which the mass is situated, whereas j is the number of the slice to which the mass belongs. Thus each mass can be designated by $M_{i,j}$.

Figure 1 represents an instance of a spider-web spring-mass network. The masses are placed in an arbitrary position. But at this stage, we make no hypothesis on the position of the masses.

1.2 The physical structure of the network

On each level $i \in [1, N]$, for each $j \in [1, M - 1]$, mass $M_{i,j}$ is linked to mass $M_{i,j+1}$ by a *transverse* spring. In addition, mass $M_{i,M}$ is linked to mass $M_{i,1}$ also by a *transverse* spring. Thus the masses on each level form closed loops.

Furthermore, on each slice $j \in [1, M]$, and for each $i \in [1, N - 1]$, mass $M_{i,j}$ is linked to mass $M_{i+1,j}$ by *longitudinal* springs. In addition, mass $M_{N,j}$ is linked to the apex mass, also by a *longitudinal* spring.

Now the topology of the network is completely specified.

We assume that longtudinal springs have a rest length of $l_{0l} = 0$, and that these springs have all the same stiffness k_l.

We assume that the transverse springs on the basis level are relatively stiff, such that the length of each of them can always be considered as equal to its rest length ${l_0}_t^1$. We make no assumption on the stiffness of the other transverse springs.

We also assume that transverse springs of level i have a rest length of ${l_0}_t^i$ where :

$$ {l_0}_t^i = {l_0}_t^1 \frac{(N + 1 - i)}{N} \tag{1} $$

where ${l_0}_t^1$ is the rest length of the basis transverse springs.

At this stage we make no assumption on the position of the masses

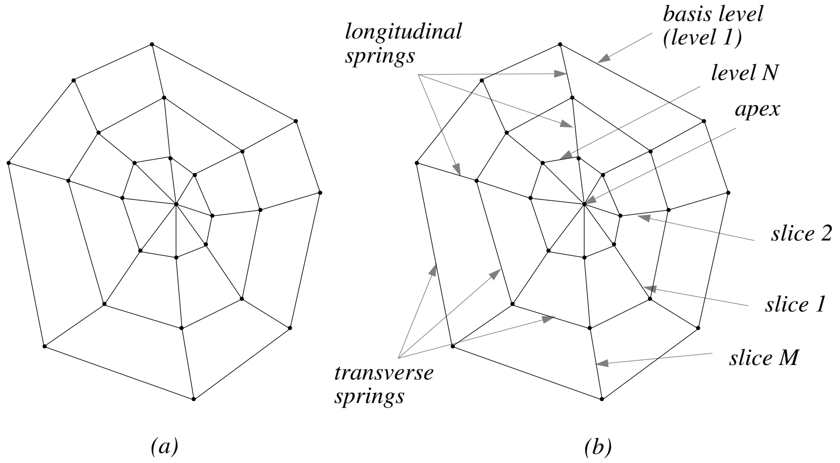

Figure 1: The structure of the spider-web spring-mass model. In this case, the number of levels is $N = 3$ and the number of slices is $M = 8$. Each mass is represented by a small dot at each vertex. These points are linked to their neighboours by *Longitudinal* springs (the radial links) and the *transverse* springs (those who form closed loops around the apex).

2 Static behaviour

In a dynamic system, Balance can be reached after a period of transient motion. The length of this period depends on the dynamic properties of the model. In this document, our objective is the study of the balance positions, regardless of any transient motion.

Dealing only with balance shapes has several important consequences. Second order mechanical systems are entirely governed by the well-known relations:

$$\begin{aligned} \vec{F} &= m\vec{a} \\ \vec{F} &= \left(k(l_0 - l) - z\frac{dl}{dt}\right)\vec{u}. \end{aligned}$$

They respectively express the acceleration $\vec{a}$ of a point of mass m on which force $\vec{F}$ is applied, and the force $\vec{F}$ produced by a damped spring of stiffness k, viscosity z, rest length l_0 and length l ($\vec{u}$ is the unit vector of the spring).

Balance implies zero velocities and zero accelerations. According to the above relations this means that the balance shape does not depend on any mass or any viscosity, but only on the links' stiffnesses and rest lengths. It should be noted that balance shape does not depend on mass considered as a coefficient of inertia, but it does depend on the weight of masses. This will be discussed in paragraph 5.2. The value of viscosity and mass strongly influence the stability of the model and the time required to reach balance. But the balance itself only depends on the links' stiffnesses and rest lengths.

2.1 Approach

Section 3 is an incremental study of this network's balance shape. In particular it shows that under specific conditions all of the levels remain parallel at rest. Section 4 is a more specific study of a case in which the basis level is constrained to be on a plane P. We prove that in this case, the volume of the spider-web spring-mass model depends only on the area of the basis level and on a wieghted sum of the heights of the $N + 1$ levels. Thus it becomes possible to obtain volume conservation with a mere sum-of-lengths conservation.

3 Incremental study of the balance shape

In the previous section, subsections 1.2 and 1.1 describe the whole model. But our approach is not to study the entire model at once. We start from a model containing only longitudinal springs, and subject to no external forces, and we

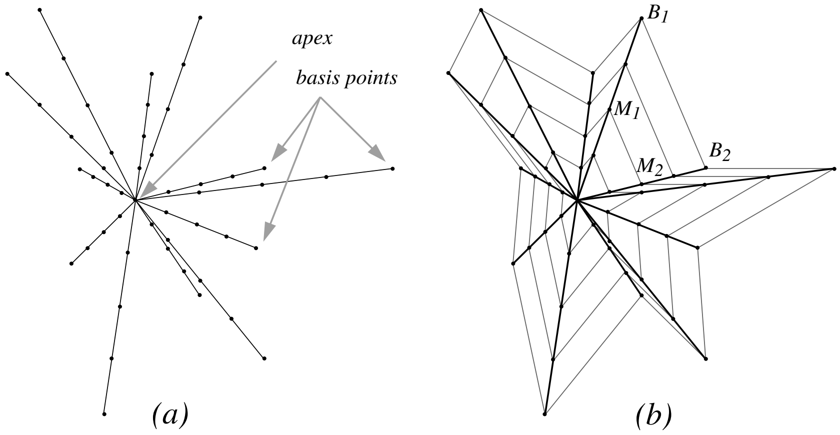

Figure 2: (a) With only longitudinal springs, the balance position of the masses on a given slice forms line segments with regularly spaced masses. In (b), we still have only longitudinal springs, but with another visualisation. We have drawn line segments in gray between the masses on a same level and on consecutive slices. It is proved that these line segments are all parallel on each slice. On this figure M_1M_2 is parallel to B_1B_2.

analytically determine its rest shape in this situation (subsection 3.1) Next we add the transverse springs, and study the affects of this addition (subsection 3.2) And last of all we exert external forces on this network and analytically study the properties of the resulting deformations (subsection 3.3).

3.1 Only longitudinal springs

3.1.1 Linear slices

We first consider the basis level. The basis points may be subject to some external forces. We make no assumption on these forces. We simply consider that they do not prevent the system from reaching a balance position $M_{1,j}$. We make no assumption on these positions. The basis level may be planar or not.

Next we add the longitudinal springs alone, without any transverse spring. This sub-model is represented on figure 2.

These springs have a zero rest length, therefore they are totally linear. In other words, if two masses A and B are linked by such a spring, the force exerted by A on B through the spring is expressed by :

$$\overrightarrow{F}_{AB} = -k_l\overrightarrow{AB} \tag{2}$$

where k_l is the stiffness of the longitudinal spring.

If the whole system is in balance, then each mass is in balance, i.e. the sum of forces exerted on a mass is zero. For every mass $M_{i,j}$ other than a basis mass and the apex ($i \neq 1 and i \neq N+1$) these external forces are composed of two forces exerted by two longitudinal springs : ($\overrightarrow{F}_{i+1,j}$, $\overrightarrow{F}_{i-1,j}$) Balance implies that

$$(\forall j \in [1, M])(\forall i \in [2, N])$$

$$\begin{aligned} & \overrightarrow{F}_{i+1,j} + \overrightarrow{F}_{i-1,j} = \vec{0} \\ \Leftrightarrow\quad & k_l\overrightarrow{M_{i,j}M_{i+1,j}} + k_l\overrightarrow{M_{i,j}M_{i-1,j}} = \vec{0} \\ \Leftrightarrow\quad & \overrightarrow{M_{i,j}M_{i+1,j}} = \overrightarrow{M_{i-1,j}M_{i,j}} = \vec{0} \end{aligned}$$

This means that $M_{i-1,j}$, $M_{i,j}$ and $M_{i+1,j}$ are aligned and that the distance between $M_{i,j}$ and both neighbours is equal.

Let us generalize this reasoning to all of the masses on a slice, except the basis masses and the apex. The above result implies that Whatever the position of the apex and whatever the position of the basis point, the other masses at rest form a line segment linking the basis point and the apex, and that the distances between two consecutive masses on this chain are equal. This situation is represented on figure 2.a.

More specifically, the rest position of each mass (other than the apex and the basis points) is determined by relation (3).

$$(\forall j \in [1, M])(\forall i \in [1, N])\overrightarrow{M_{i,j}M_{N+1}} = \frac{N+1-i}{N}\overrightarrow{M_{1,j}M_{N+1}} \tag{3}$$

where M_{N+1} is the position of the apex point.

3.1.2 The position of the apex point

Now let us express the fact that the apex point is in balance. The only external forces applied on the apex point are the M forces applied by the masses on the N-th level. Balance means :

$$\sum_{j=1}^{M} \overrightarrow{F}_{N,N+1} = \vec{0} \tag{4}$$

$$\Leftrightarrow \quad k_l \sum_{j=1}^{M} \overrightarrow{M_{N+1}M_{N,j}} = \vec{0} \tag{5}$$

$$\tag{6}$$

On the other hand equation (3) implies that

$$(\forall j \in [1, M])\overrightarrow{M_{N+1}M_{N,j}} = \frac{1}{N}\overrightarrow{M_{N+1}M_{1,j}} \tag{7}$$

The combination of both equations yeilds :

$$k_l \sum_{j=1}^{M} \overrightarrow{M_{N+1}M_{1,j}} = \vec{0} \tag{8}$$

$$\Leftrightarrow \quad \overrightarrow{M_{N+1}} = \frac{1}{M}\sum_{j=1}^{M} \overrightarrow{M_{1,j}} \tag{9}$$

This means that whatever the balance position $M_{1,j}$ of the basis points, the balance position of the apex point is the barycentre of the basis points.

3.1.3 Similarity between the N levels

Let us study the relative position $\overrightarrow{M_{i,j}M_{i,j+1}}$ of two masses on the same level i and on neighbouring slices j and $j+1$. According to equation (3) :

$$\begin{aligned}\overrightarrow{M_{i,j}M_{N+1}} &= \frac{N+1-i}{N}\overrightarrow{M_{1,j}M_{N+1}}\\ \overrightarrow{M_{i,j+1}M_{N+1}} &= \frac{N+1-i}{N}\overrightarrow{M_{1,j+1}M_{N+1}}\end{aligned}$$

Substracting the two equations yields :

$$\overrightarrow{M_{i,j}M_{i,j+1}} = \frac{N+1-i}{N}\overrightarrow{M_{1,j}M_{1,j+1}} \tag{10}$$

This means that, whatever the balance positions $M_{1,j}$ and $M_{1,j+1}$ of basis points on two consecutive slices, the line segments linking $M_{i,j}$ and $M_{i,j+1}$, the masses between the same two slices but on any other level i, are parallel to the basis segment $M_{1,j}M_{i,j+1}$. (cf figure 2.b) Drawing lines between the masses on the same level produces a polygon. (cf figure 2.b) On the basis level, we have the *basis polygon*. On other level it is called the *i-th level polygon*. Equation (10) implies that for any i, the $i - thlevelpolygon$ is geometrically *similar* to the basis polygon. In other words, the $i - th$ polygon can always be obtained from the basis polygon by a scaling operation of centre M_{N+1} (the apex point) and of scaling factor $(N + 1 - i)/N$.

3.1.4 Conclusion

The conclusion of the previous paragraphs is that, whatever the balance position $M_{i,j}$ of the basis points, depending on the forces exerted by the network and other external forces, the balance position of the apex point is the barycentre of the basis points (cf equation (9)). The other masses of a given slice j at balance form linear chains linking the apex and the basis points. In these chains, the distances between consecutive masses of a slice are equal.

Moreover the polygons defined by the masses of a given level are all geometrically similar to the polygon defined by the masses of the basis level.

In other words, the balance positions of the masses on each level can be determined with a simple scaling operation from the positions of the basis level.

3.2 Adding transverse springs

Before adding all of the transverse springs, let us only add the basis transverse springs. Let us also take in account the fact that the basis transverse springs are relatively stiff, such that the lengths of these springs at balance, always equal their rest length ${l_0}_t^1$.

In these conditions, according to equation (10), the length of an edge of the i-th level polygon is also constant and always equals ${l_0}_t^1(N+1-i)/N$.

According to equation (1), the transverse spring of level i has precisely a rest length of ${l_0}_t^i = {l_0}_t^1(N+1-i)/N$. In these conditions, even if the transverse springs have a great influence on the dynamics of the system, they have absolutely no influence on the balance position of the network. Therefore, whatever the stiffness of the transverse springs, the conclusions of the last section (without trasnverse springs) are still true in the presence of transverse springs, provided that their rest length equals ${l_0}_t^i = {l_0}_t^1(N+1-i)/N$.

In the same way, as long as the rest distance between the masses on the same level equals ${l_0}_t^1(N+1-i)/N$, the balance position with and without the transverse springs will be identical. So in the rest of this document, we will continue working on a model with no transverse springs, but at each stage, we will show that the rest distances between the masses of the same level remain unchanged, and therefore that the rest shape will remain the same in the presence of the transverse springs.

3.3 Exerting forces on the network

3.3.1 Simpler system : a mass-spring chain

The model at this stage is simply a set of mass-spring chains. So let us study the behaviour of a simple mass-spring chain composed of $N+1$ masses, in which the springs have a zero rest length and stiffness k_l. Let M_1 be position of one end and M_{N+1} be the position of the other end. Between these ends, the position of the i-th mass is M_i.

Furthermore, we consider that $(\forall i \in [2,N])$ a force $\overrightarrow{F}_i$ is exerted on the i-th mass. This force is constant in the course of time (otherwise there would be no balance position). We make no assumption on the intensity and the direction of these forces.

Our objective is to calculate the balance position of all of the chain's masses as a function of the applied forces and as a function of the balance positions of the chain's ends M_1 and M_{N+1}.

3.3.2 A linear system

Let us consider the i-th mass. Its balance position is M_i and it is subject to force $\overrightarrow{F}_i$. The balance condition on this i-th mass is expressed as :

$$(\forall i \in [2,N])\,\overrightarrow{F}_i = -k_l(\overrightarrow{M_iM_{i+1}} + \overrightarrow{M_iM_{i-1}}) = k_l(2\overrightarrow{M_i} - \overrightarrow{M_{i-1}} - \overrightarrow{M_{i+1}}) \tag{11}$$

For all masses, this yields a matrix equation :

$$\begin{pmatrix} \overrightarrow{F}_2 \\ \\ \vdots \\ \\ \overrightarrow{F}_N \end{pmatrix} = k_l \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & -1 & \ddots & & 0 \\ \vdots & & & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} \overrightarrow{M}_2 \\ \\ \vdots \\ \\ \overrightarrow{M}_N \end{pmatrix} - k_l \begin{pmatrix} \overrightarrow{M}_1 \\ 0 \\ \vdots \\ 0 \\ \overrightarrow{M}_{N+1} \end{pmatrix} \tag{12}$$

$$\Leftrightarrow [F] = k_l\,[A]\,[M] - k_l[Ends] \tag{13}$$

where the unknown is the $[M]$ matrix, i.e. the $\overrightarrow{M}_2 \dots \overrightarrow{M}_N$ position vectors. $[A]$ is a symmetrical real matrix. Therefore it is invertible. Let $\left[A^{-1}\right]$ be the inverse matrix.

$$[M] = \frac{1}{k_l}\left[A^{-1}\right][F] + \left[A^{-1}\right][Ends] \tag{14}$$

In equation (14) the first term represents the deformation of the chain, and the second term represents its rest shape $([F] = 0)$. We are not interested in the value of this inverse matrix. The most important point to be noted is that the rest shape depends on the position of the chain's ends, but that the deformation does not (!) The deformation only depends on $\left[A^{-1}\right]$ and on k_l.

As a conclusion, for a spring-mass chain, whose springs have zero rest lengths, the deformations caused by the application of a set of forces does not depend on the position of the end points. For example, if the displacement of the i-th mass at the application of $[F]$ is $\overrightarrow{\Delta M_i}$, then for any other position of the end points, the application of $[F]$ would displace the i-th mass in the same manner $\overrightarrow{\Delta M_i}$.

3.3.3 Physical matrix inversion

This paragraph is only a *by-the-way* remark on one of the interesting consequences of equation (14).

First let us note that the displacement of *one* point can be represented by a single vector (e.g. $\overrightarrow{\Delta M_i}$). But the deformation of the whole chain, whether it is caused by one force (e.g. $\overrightarrow{F}_i$) or by a set of forces exerted on all of the masses ($[F]$) must be expressed by a whole vector matrix ($[M]$).

The fact that the relation between the positions $[M]$ and the forces $[F]$ applied on the masses is linear, enables us to use the *superposition theorem* and to determine the $\left[A^{-1}\right]$ matrix with only simple physical calculations.

Let us consider the displacement $[M_i^\epsilon]$ of the chain due to the application of a single force $\overrightarrow{F}_i$. More generally, for all i in $[2, N]$, $[M_i^\epsilon]$ is the displacement of the chain at the application of a single force $\overrightarrow{F}_i$ on the i-th mass. The chain is a linear system. And this is why the displacement $[M]$ due to the application of these forces all at the same time is the sum of the displacements $\left[M_j^\epsilon\right]$ due to the application of each individual force.

$$[M] = \sum_{i=1}^{N} [M_i^\epsilon] \tag{15}$$

The approach of the superposition theorem is the converse. According to this theorem, the problem of calculating the shape $[M]$ of the chain due to the application of a complete force profile $[F]$, boils down to the calculation of N separate shapes $[M_i^\epsilon]$ caused by the application of an individual force on the i-th mass. In our case, the calculation of one elementary shape $[M_i^\epsilon]$ as a function of i is an easy task. Next, it suffices to add these N separate shapes to obtain the final shape $[M]$.

It should be noted that the calculation of $[M]$ for a given set of forces $[F]$ is the same thing as calculating the inverse $\left[A^{-1}\right]$ of matrix $[A]$.

The only difference is that, in order to invert this matrix, instead of using mathematical operations, we use one simple physical calculation consisting of calculating $[M_i^\epsilon]$ as a function of i and the input force $\overrightarrow{F}_i$.

3.3.4 Application to the spider-web network

The spider-web network (with only the longitudinal springs) can be considered as a set of M spring-mass chains, whose rest lengths are zero. One end of the mass chain is the mass on the basis level and the other end is the apex mass. These M chains are exactly identical. Their only difference is the position of their ends.

As a consequence of the results of section 3.3.2, if the same force vector $\overrightarrow{F}_i$ is applied on all of the masses on the i-th level, then the displacement of all of the masses on the j-th level $(i = j or i \neq j)$ is the same.

As a conclusion, whatever the set of forces applied on the spider-web network without transverse springs, if the forces applied on the masses on the same level are equal, then the displacements of these masses are also equal.

In other words, let us consider a specific balance position in which the masses on each level form a specific shape. Now if a set of forces is applied on the network such as the forces applied on the masses on each level are equal then, in the resulting balance position, the levels are simply translated by the same vector. Each level may be translated by a different vector, but within the same level the translation vectors are the same. This means that the *shape* of each level does not change by the application of such forces. (figure 3)

In particular this means that if the distances between the masses on the i-th level equal ${l_0}_t^1(N+1-i)/N$, then after the application of such forces, the rest distances will still equal ${l_0}_t^1(N+1-i)/N$.

Thus once again, the rest shape with and without the tranverse springs is the same.

3.3.5 Conclusion

As it is also mentioned in [1]

We have just *proved* that if:

1. the longitudinal springs have zero rest length
2. the transverse basis springs are quasi-rigid
3. the rest lengths of the transverse springs are defined by ${l_0}_t^i = {l_0}_t^1 \frac{(N+1-i)}{N}$
4. all other forces applied to the model have the same value, on the masses of each level,

then the balance shape produced by the points on each level i can be obtained from the balance shape of the basis points by a simple translation and a scaling by a factor of $(N+1-i)/N$.

This translation depends on the applied forces and cannot be determined analytically. This is why it is interesting to use dynamic simulation. However, this result allows us to conclude that in such cases, a simple sum-of-lengths-conservation operation can provide volume conservation.

4 The volume of the spider-web spring-mass network

A spring-mass network is a set of points. The springs may define a point-to-point neighbourhood. But this may only define edges. Objectively speaking, the notions of surface and volume are not inherent in a spring-mass network, and need to be explicitly defined. Subsection 4.1 defines, in a particular case, what the volume of a spider-web network may be, and subsection determines this volume.

4.1 The definition of the spider-web's volume

4.1.1 First the notion of surface

As it is expressed in relation (10), if the forces applied on the masses of each level are equal, then, on a given slice j, all of the edges of the i-th level polygon are parallel. More specifically, whatever levels i_1 and i_2, line segments $M_{i_1,j}M_{i_1,j+1}$ and $M_{i_2,j}M_{i_2,j+1}$ are always parallel at rest. In particular, this is true for consecutive levels i and $i+1$. This implies that, at rest, $(M_{i,j}M_{i+1,j}M_{i+1,j+1}M_{i,j+1})$ is always a planar polygon (figure 3).

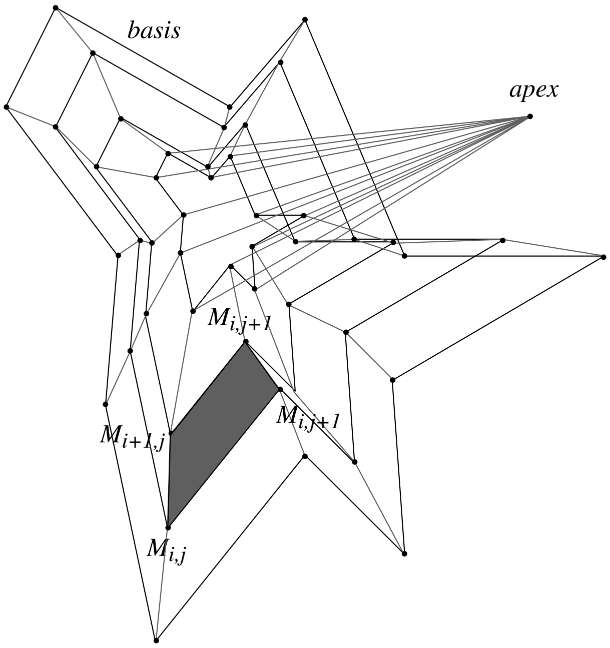

Figure 3: The same spider-web network as on figure 2 but with transverse springs and with different external forces applied on each level. Each level has been displaced without any deformation. Despite all this deformation, $M_{i,j}M_{i,j+1}$ and $M_{i+1,j}M_{i+1,j+1}$ are parallel line segments. This implies that $(M_{i,j}M_{i+1,j}M_{i+1,j+1}M_{i,j+1})$ (represented in gray) is always a planar polygon.

Let S_{tot} be the union of the following set of polygons :

$$S_{tot} = \left(\bigcup_{\substack{i \in [1, N-1] \\ j \in [1, M-1]}} (M_{i,j}M_{i+1,j}M_{i+1,j+1}M_{i,j+1}) \cup \bigcup_{j \in [1, M-1]} (M_{N,j}M_{N,j+1}M_{N+1}) \right) \tag{16}$$

S_{tot} is a surface composed of polygons and which contains all of the masses and edges of the spider-web network.

Intuitively, if this network was to be considered as a discrete representation of a continuous surface, S_{tot} would probably be one possible continuous surface. Moreover, for simple cases, as those represented on the figures of this document, the spider-web network seemed to be objectively composed of *tiles*. These *tiles* are the components of the union operation in relation (16). And S_{tot} is no more than the union of these tiles.

Planar polygons (and triangles in particular) can always be characterized by one and only one area. S_{tot} is exclusively composed of planar polygons. Therefore S_{tot} can always be characterized by one and only one total area A_{tot}.

As a conclusion, now, the spider-web network can be characterized by a surface, which is the surface of S_{tot}.

4.1.2 Planar basis

In the previous section, we made no assumption on the forces applied on the basis masses or on the resulting balance positions. In this section, we assume that all of the basis points are constrained to move on a plane called P.

Furthermore let us orient this plane. In other words, let us distinguish what is *above* the plane (positive heights) and what is *underneath* the plane (negative heights).

In this case, if the external forces on the masses of each level are equal, then the previous section proves that, at rest, the i-th level polygon can be obtained from the basis polygon by one translation and one scaling operation.

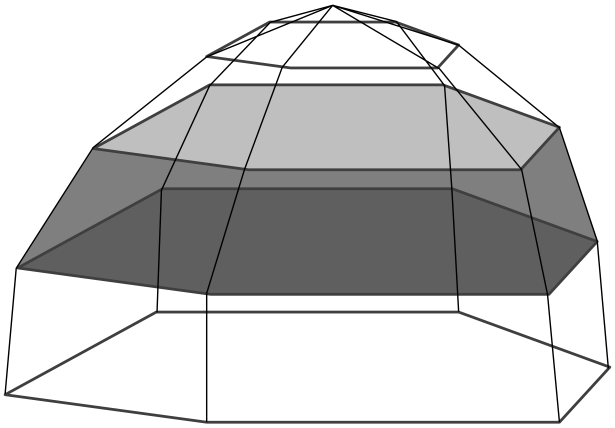

Figure 4: A spider-web mass-spring network with 5 levels and 6 slices. The volume V_2 between the second level and the third level has been shaded in gray.

This implies that if the basis polygon is planar, then all of the other polygons are also planar and parallel to the basis polygon. As a consequence, the i-th level polygon is always parallel to the $i+1$-th level polygon.

This implies that each level $i \in [1, N+1]$ can be characterized by one height value h_i from the P plane. In particular $h_1 = 0$ since the basis polygon is constrained to lie on the P plane.

4.1.3 No self intersections

We also assume that the edges of the basis level do not intersect each other (except on shared vertices) In this case, the basis level can be characterized by an area A_1. The i-th level polygon ($\forall i \in [1, N]$) is similar to the basis polygon, therefore it does not have intersecting edges either. Therefore the i-th level polygon can also be characterized by an area A_i.

Finally this implies that the i-th level polygon and the $i+1$-th level polygon, and the M polygons of S_tot which link these two levels enclose a connexe volume V_i. Cf figure 4. On this figure, whatever $i \in [1, 4]$ the i-th level is closer to the plane than the $i+1$-th level. This is not necessarily the case. We arbitrarily decide that if this is not the case, then the corresponding volume will be negative.

We define the volume V_{tot} of the spider-web network as the sum of the volumes of each level :

$$V_{tot} = \sum_{i=0}^{N} V_i \tag{17}$$

4.2 The calculation of V_{tot}

There are straightforward methods to calculate the volume of an arbitrary object composed of polygons. But our objective is to calculate this volume as a function of the heights h_i.

4.2.1 Dividing each level in simple parts

The surface of a planar polygon with n edges can always be divided in n triangles. It suffices to link all of the polygon's vertices to an arbitrary point O on the same plane as the polygon's vertices. For the sake of simplicity, we choose to place O inside the polygon, such that the area of the polygon is the sum of the areas of the n triangles.

Let us perform this division on the i-th level polygon, with a central point called O_i (Cf figure 5) Next we know that the i-th level polygon and the $i+1$-th level polygon are similar. This means that there exists a scaling operation with a specific center O such that the application of this operation on the i-th level polygon yields the $i+1$-th level polygon. This implies that for all $j \in [1, M]$, the $M_{i,j}Mi+1, j$ lines all intersect at O. The intersection of the (OO_i) line and the $i+1$-th level polygon will be called O_{i+1}.

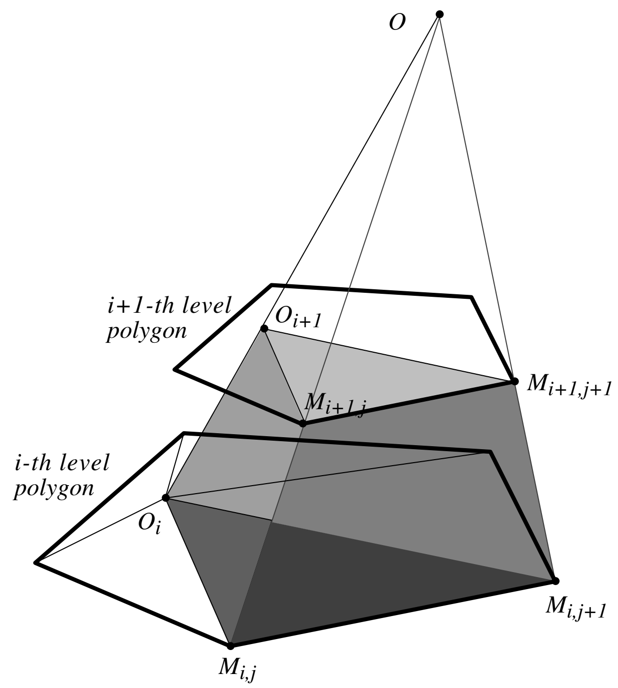

Figure 5: A part of a 5-sliced spider-web network. Only the i-th level and the $i+1$-th level have been represented. The truncated tetrahedron T_j has been shaded in gray.

With O_i and O_{i+1} and the vertices of two consecutive slices j and $j+1$, we can build a truncated tetrahedron T_j. (cf figure 5). This can be repeated for all of the other slices. Thus we obtain M truncated tetrahedra.

The volume V_i between the i-th level polygon and the $i+1$-th level polygon is the sum of the volumes of the M truncated tetrahedra. Thus the calculation of the V_i volume boils down to the calculation of the volumes of the M truncated tetrahedra.

4.2.2 The calculation of V_i

Let h be the height of a truncated tetrahedron. Let A be the area of the larger basis and let $\alpha^2 A$ be the area of the smaller basis ($\alpha \in [0,1]$). In this case, the volume V of the truncated tetrahedron equals :

$$V = \frac{1}{3}Ah(1+2\alpha) \tag{18}$$

In the case of the spider-web network, the scaling factor between two successive levels is :

$$\alpha = \frac{\frac{N-i}{N}}{\frac{N+1-i}{N}} = \frac{N-i}{N+1-i} \tag{19}$$

As a conclusion, the volume V_i between the i-th level polygon and the $i+1$-th level polygon is :

$$V_i = \frac{1}{3}A_i\left(h_{i+1}-h_i\right)(1+2\alpha_i) = \frac{1}{3}A_i\left(h_{i+1}-h_i\right)\left(1+2\frac{N+1-i}{N}\right) \tag{20}$$

But the relation between the area A_i of the i-th level polygon and the area A_1 of the basis level polygon is the square of the scaling factor between both levels. Therefore :

$$A_i = \left(\frac{N+1-i}{N}\right)^2 A_1 \tag{21}$$

As a conclusion :

$$V_i = \frac{1}{3} A_1 \left(h_{i+1} - h_i\right) \frac{(N+1-i)(3N-3i+1)}{N^2} \tag{22}$$

4.2.3 V_tot

The addition of the V_i terms yields :

$$V_{tot} = \frac{1}{3N^2} A_1 \sum_{i=1}^{N+1} \beta_i h_i \tag{23}$$

where the β_i are constant parameters depending only on the value of N.

As a conclusion, the total volume of the spider-web network is the product of A_1 (the area of the basis polygon) and a weighted sum of the heights of the polygons on different levels.

5 Application to the deformation of droplets

The aim of this section is not to present a detailed discussion on droplets. This has already been developed in [1]. Our aim is simply to link both studies.

5.1 Approach

We assume that the shape of the droplets is determined by three global constraints :

a The drop has a constant volume

b Due to capillarity forces, the contact surface with the air is the smallest possible.

c If the drop lies on a hydrophilic surface, its contact surface with this surface is the largest possible

Furthermore, various types of external forces may apply on droplets (e.g. gravity).

The resolution of constaint a and b produces trivially a sphere. The resolution of constraints a, b and c generally produces more complex shapes.

By using the results of the analytical proofs provided in this document, [1] show that it is possible to use a spider-web spring-mass model to satisfy these constraints.

In order to satisfy constraint c we use M masses which are contrained to move upon a plane P. This plane represents the hydrophilic surface and the masses represent the limits of the contact surface between the droplet and the hydrophilic surface. In order to maximize the area of this surface, we join these M masses by springs with very large rest lengths.

Next, in order to satisfy constraint b, we model the contact surface between the the drop and the air by a *spider-web spring-mass* network. More specifically, we consider the M points moving on the plane as the basis points of a spider-web network. The fact that the longitudinal springs have zero rest lengths tends indeed to reduce the contact surface between the air and the drop. But if we stop at this stage, the spider-web network is flat due to its tension. It has no volume.

Now we should satisfy constraint a which specifies that the volume of the drop is constant. According to the conclusions of section 4, the volume of the drop model is the product of the area of the drop's basis and a weighted sum of the heights of the drop's different levels.

In practice we satisfy constraint a with a special physical interaction composed of a set of links whose weighted sum-of-lenghts at rest is constant. (The weights are the β_i factors mentioned in relation (23).) This interaction as well as gravity and other external forces exert the same forces on the masses of the same level. Therefore in all these cases, the results of section 2 remain valid.

Thus it is analytically *proved* that the weighted sum of heights of the droplet model at rest remains constant, and that if the area of the basis remains constant then the rest volume of the drop is always the same. This *proved* fact combined with the *experimentally observed* fact that the area of basis does tend to remain constant yeilds practical conclusion that the rest volume of the drop remains constant.

By adjusting the stiffness of the basis springs, the longitudinal springs or those of the sum-of-lengths-constant interaction, it is possible to control to which extent contraints a, b and c are satisfied, and thus obtain different types of drops.

5.2 The link between both models

Let us finally note that in this physical context, the properties of the spider-web network meet physical justifications. Indeed for the sake of symmetry, masses and stiffnesses on each level are the same. However, if the spider-web must represent a continuous reference surface, and if the density and surface tension of this continuous surface are to be uniform, then mass values and transverse stiffnesses must *not* be equal on different levels.

In order to have uniform density, masses on level i must be all the smaller than the average perimeter of the i-th level polygon is small.

In order to have uniform tension, the stiffness of transverse springs must be all the smaller than the average perimeter of the i-th level polygon is small. More generally the stiffness of any spring must be all the smaller than its average length is small.

The average shape of this model is a hemisphere. Therefore the average length of the longitudinal springs is the same for all slices and all levels. On the contrary, the average length of the transverse springs grows smaller as the levels are close to the apex.

At last let us point out another consequence of the equality of masses on each level. Due to this fact gravity exerts the same force on the masses of the same level. Thus the results of section 2 remain valid. We consider gravity as a mere external force field. In this way, the role of *mass* considered as a coefficient of inertia is clearly distinguished from the role of *mass* considered as a gravitational coefficient.

References

[1] P. Fournier, P. Poulin, and A. Habibi. Simulating the flow of liquid droplets. In *Graphics Interface 98*, page ??, May 1998.